

# Counting and packing Hamilton cycles in dense graphs and oriented graphs

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## Abstract

We present a general method for counting and packing Hamilton cycles in dense graphs and oriented graphs, based on permanent estimates. We utilize this approach to prove several extremal results. In particular, we show that every nearly  $cn$ -regular oriented graph on  $n$  vertices with  $c > 3/8$  contains  $(cn/e)^n(1 + o(1))^n$  directed Hamilton cycles. This is an extension of a result of Cuckler, who settled an old conjecture of Thomassen about the number of Hamilton cycles in regular tournaments. We also prove that every graph  $G$  on  $n$  vertices of minimum degree at least  $(1/2 + \varepsilon)n$  contains at least  $(1 - \varepsilon)\text{reg}_{\text{even}}(G)/2$  edge-disjoint Hamilton cycles, where  $\text{reg}_{\text{even}}(G)$  is the maximum *even* degree of a spanning regular subgraph of  $G$ . This establishes an approximate version of a conjecture of Kühn, Lapinskas and Osthus.

## 1 Introduction

A *Hamilton cycle* in a graph or a directed graph is a cycle passing through every vertex of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity is one of the most central notions in graph theory, and has been intensively studied by numerous researchers. Since the problem of determining Hamiltonicity of a graph is NP-complete it is important to find general sufficient conditions for Hamiltonicity and in the last 60 years many interesting results were obtained in this direction. Once Hamiltonicity is established it is very natural to strengthen such result by showing that a graph in question has many distinct or edge-disjoint Hamilton cycles.

In this paper we present a general approach for counting and packing Hamilton cycles in dense graphs and oriented graphs. This approach is based on the standard estimates for the permanent of a matrix (the famous Minc and Van der Waerden conjectures, established by Brégman, and by Egorychev and by Falikman, respectively). In a nutshell, we use these permanent estimates to show that an  $r$ -factor in a given graph or digraph  $G$  on  $n$  vertices, where  $r$  is linear in  $n$ , contains many (edge-disjoint)

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2-factors in the undirected case or 1-factors in the directed case, whose number of cycles is relatively small (much smaller than linear); then these factors are converted into many (edge-disjoint) Hamilton cycles using rotation-extension type techniques. Strictly speaking, the permanent-based approach to Hamiltonicity problems is not exactly new and has been used for the first time in [12] and then in several subsequent papers ([18], [20], [19], [13]). However, these prior papers worked in the setting of random or pseudo-random graphs, while the present contribution appears to be the first one where the permanent-based approach is applied in the general, extremal graph theoretic setting.

We employ our method to prove several new extremal results and to derive some known results in a conceptually different and easier way as well.

One of the first and probably most celebrated sufficient conditions for Hamiltonicity was established by Dirac [8] in 1952, who proved that every graph on  $n$  vertices,  $n \geq 3$ , with minimum degree at least  $n/2$  is Hamiltonian. The complete bipartite graph  $K_{m,m+1}$  shows that this theorem is best possible, i.e., the minimum degree condition cannot be improved. Later, Nash-Williams [26] proved that any *Dirac graph* (that is, a graph  $G$  on  $n$  vertices with minimum degree  $\delta(G) \geq n/2$ ) has at least  $\frac{5}{224}n$  edge-disjoint Hamilton cycles. He also asked [25, 26, 27] to improve this estimate. Clearly,  $\lfloor (n+1)/4 \rfloor$  is a general upper bound on the number of edge-disjoint Hamilton cycles in a Dirac graph and originally Nash-Williams [25] believed that this is tight.

Babai (see also [25]) found a counterexample to this conjecture. Extending his ideas further, Nash-Williams gave an example of a graph on  $n = 4k$  vertices with minimum degree  $2k$  and with at most  $\lfloor (n+4)/8 \rfloor$  edge-disjoint Hamilton cycles. He conjectured that this example is tight, i.e., any Dirac graph contains at least  $\lfloor (n+4)/8 \rfloor$  edge-disjoint Hamilton cycles. Moreover, Nash-Williams pointed out that the example depends heavily on the graph being not regular. He thus also proposed the following conjecture which has become known as the “Nash-Williams Conjecture”:

**Conjecture 1.1** *Every  $d$ -regular Dirac graph contains  $\lfloor d/2 \rfloor$  edge-disjoint Hamilton cycles.*

Recently, this conjecture was settled asymptotically by Christofides, Kühn and Osthus [5], who proved that any  $d$ -regular graph  $G$  on  $n$  vertices with  $d \geq (1/2 + \varepsilon)n$ , contains at least  $(1 - \varepsilon)d/2$  edge-disjoint Hamilton cycles. For large graphs, Kühn and Osthus [23] further improved it to  $\lfloor d/2 \rfloor$  edge-disjoint Hamilton cycles.

For the non-regular case, Kühn, Lapinskas and Osthus [21] proved that if  $\delta(G) \geq (1/2 + \varepsilon)n$ , then  $G$  contains at least  $\text{reg}_{\text{even}}(n, \delta(G))/2$  edge-disjoint Hamilton cycles where  $\text{reg}_{\text{even}}(n, \delta)$  is the largest *even* integer  $r$  such that every graph  $G$  on  $n$  vertices with minimum degree  $\delta(G) = \delta$  must contain an  $r$ -regular spanning subgraph (an  $r$ -factor). As for a concrete  $G$ , the maximal *even* degree  $r$  of an  $r$ -factor of  $G$ , which we denote by  $\text{reg}_{\text{even}}(G)$ , can be much larger than  $\text{reg}_{\text{even}}(n, \delta)$ . Therefore, it is natural to look for bounds in terms of  $\text{reg}_{\text{even}}(G)$ . In [23], Kühn and Osthus showed that any graph  $G$  with  $\delta(G) \geq (2 - \sqrt{2} + \varepsilon)n$  contains  $\text{reg}_{\text{even}}(G)/2$  edge-disjoint Hamilton cycles, and in [21], Kühn, Lapinskas and Osthus conjectured the following tight result.

**Conjecture 1.2** *Suppose  $G$  is a Dirac graph. Then  $G$  contains at least  $\text{reg}_{\text{even}}(G)/2$  edge-disjoint Hamilton cycles.*

Answering an open problem from [21], in this paper we prove this conjecture asymptotically.

**Theorem 1.3** *For every  $\varepsilon > 0$ , there exists  $n_0 = n(\varepsilon)$  so that for every graph  $G$  with  $|V(G)| = n \geq n_0$  and  $\delta(G) \geq (1/2 + \varepsilon)n$ ,  $G$  contains at least  $(1 - \varepsilon)\text{reg}_{\text{even}}(G)/2$  edge-disjoint Hamilton cycles.*

Given a graph  $G$ , let  $h(G)$  denotes the number of distinct Hamilton cycles in  $G$ . Strengthening Dirac's theorem Sárközy, Selkow and Szemerédi [30] proved that every Dirac graph  $G$  contains not one but at least  $c^n n!$  Hamilton cycles for some small positive constant  $c$ . They also conjectured that  $c$  can be improved to  $1/2 - o(1)$ . This has later been proven by Cuckler and Kahn [7]. In fact, Cuckler and Kahn proved a stronger result: every Dirac graph  $G$  on  $n$  vertices with minimum degree  $\delta(G)$  has  $h(G) \geq \left(\frac{\delta(G)}{e}\right)^n (1 - o(1))^n$ . The random graph  $G(n, p)$  with  $p > 1/2$  shows that this estimate is sharp (up to the  $(1 - o(1))^n$  factor). Indeed in this case with high probability  $\delta(G(n, p)) = pn + o(n)$  and the expected number of Hamilton cycles is  $p^n(n - 1)! < (pn/e)^n$ .

To illustrate our techniques we prove the following theorem which gives a lower bound on the number of Hamilton cycles in a dense graph  $G$  in terms of  $\text{reg}(G)$ . Although this bound is asymptotically tight for nearly regular graphs, it is weaker than the result of Cuckler and Kahn in general. On the other hand, since every Dirac graph contains an  $r$ -factor with  $r$  about  $n/4$  (see [16]), our bound implies the result of Sárközy, Selkow and Szemerédi mentioned above.

**Theorem 1.4** *Let  $n$  be a sufficiently large integer and let  $G$  be a graph on  $n$  vertices such that  $\delta(G) \geq n/2$ . Then the number of Hamilton cycles in  $G$  is at least  $\left(\frac{\text{reg}(G)}{e}\right)^n (1 - o(1))^n$ .*

Theorem 1.4 implies that, given a dense regular graph  $G$ , the number of Hamilton cycles in  $G$  is asymptotically exactly (in exponential terms) what we expect in a random graph with the same edge density.

**Corollary 1.5** *For every  $c \geq 1/2$  there exists an integer  $n_0 := n_0(c)$  such that the following holds. Suppose that*

- (i)  $G$  is a graph with  $|V(G)| = n \geq n_0$ , and
- (ii)  $G$  is a  $cn$ -regular graph.

*Then  $h(G) = \left(\frac{cn}{e}\right)^n (1 + o(1))^n$ .*

Using a technical lemma from [5], in Section 2 we show that given an almost regular graph  $G$  on  $n$  vertices with  $\delta(G) \geq n/2 + \varepsilon n$ ,  $G$  contains an  $r$ -factor with  $r$  very close to  $\delta(G)$ . Therefore, we conclude that if the minimum degree of  $G$  is at least  $n/2 + \varepsilon n$ , then condition (ii) in Corollary 1.5 can be relaxed to the requirement that  $G$  is “almost regular”.

**Corollary 1.6** *For every  $c > 1/2$  there exist  $\varepsilon > 0$  and an integer  $n_0 := n_0(c, \varepsilon)$  such that the following holds. Suppose that:*

(i)  $G$  is a graph with  $|V(G)| = n \geq n_0$ , and

(ii)  $d(v) = (c \pm \varepsilon)n$  for every  $v \in V$ .

Then  $h(G) = \left(\frac{(c \pm \varepsilon')n}{e}\right)^n$  where  $\varepsilon'$  tends to 0 with  $\varepsilon$ .

An *oriented* graph  $G$  is a graph obtained by orienting the edges of a simple graph. That is, between every unordered pair of vertices  $\{x, y\} \subseteq V(G)$  there exists at most one of the (oriented) edges  $xy$  or  $yx$ . Hamiltonicity problems in oriented graphs are usually much more challenging. Given an oriented graph  $G$ , let  $\delta^+(G)$  and  $\delta^-(G)$  denote the minimum *outdegree* and *indegree* of the vertices in  $G$ , respectively. We also set  $\delta^\pm(G) = \min\{\delta^+(G), \delta^-(G)\}$ . A *semi-degree* of an oriented graph  $G$  is just the parameter  $\delta^\pm(G)$ . In the late 70's Thomassen [33] raised the natural question of determining the minimum semi-degree that ensures the existence of a Hamilton cycle in an oriented graph  $G$ . Häggkvist [14] found a construction which gives a lower bound of  $\frac{3n-4}{8} - 1$ . The problem was resolved only recently by Keevash, Kühn and Osthus [17], who proved that every oriented graph  $G$  on  $n$  vertices with  $\delta^\pm(G) \geq \frac{3n-4}{8}$  contains a Hamilton cycle.

Counting Hamilton cycles in tournaments is another very old problem which goes back some seventy years to one of the first applications of the probabilistic method by Szele [32]. He proved that there are tournaments on  $n$  vertices with at least  $(n-1)!/2^n$  Hamilton cycles. Alon [1] showed that this result is nearly tight and every  $n$  vertex tournament has at most  $O(n^{3/2}(n-1)!/2^n)$  Hamilton cycles. Thomassen [34] and later Friedgut and Kahn [11] conjectured that the randomness is unnecessary in Szele's result and that in fact every regular tournament contains at least  $n^{(1-o(1))n}$  Hamilton cycles. This conjecture was solved by Cuckler [6] who proved that every regular tournament on  $n$  vertices contains at least  $\frac{n!}{(2+o(1))^n}$  Hamilton cycles. The following theorem substantially extends Cuckler's result [6].

**Theorem 1.7** *For every  $c > 3/8$  and every  $\eta > 0$  there exist a positive constant  $\varepsilon := \varepsilon(c, \eta) > 0$  and an integer  $n_0 := n_0(c, \eta)$  such that the following holds. Suppose that:*

(i)  $G$  is an oriented graph with  $|V(G)| = n \geq n_0$ , and

(ii)  $d^\pm(v) = (c \pm \varepsilon)n$  for every  $v \in V(G)$ .

Then  $h(G) = \left(\frac{(c \pm \eta)n}{e}\right)^n$ . In particular, if  $G$  is  $cn$ -regular, then  $h(G) = \left(\frac{(c+o(1))n}{e}\right)^n$ .

The bound on in/out-degrees in this theorem is tight. This follows from the construction of Häggkvist [14] (mentioned above), which shows that there are  $n$ -vertex oriented graphs with all in/outdegrees  $(3/8 - o(1))n$  and no Hamilton cycles.

**Definitions and notation:** Our graph-theoretic notation is standard and follows that of [35]. For a graph  $G$ , let  $V = V(G)$  and  $E = E(G)$  denote its sets of vertices and edges, respectively. For subsets  $U, W \subseteq V$ , and for a vertex  $v \in V$ , we denote by  $E_G(U)$  all the edges of  $G$  with both endpoints in  $U$ ,

by  $E_G(U, W)$  all the edges of  $G$  with one endpoint in  $U$  and one endpoint in  $W$ , and by  $E_G(v, U)$  all the edges with one endpoint being  $v$  and one endpoint in  $U$ . We write  $N_G(v)$  for the neighborhood of  $v$  in  $G$  and  $d_G(v)$  for its degree. For an oriented graph  $G$  we write  $uv$  for the edge directed from  $u$  to  $v$ . We denote by  $N_G^+(v)$  and  $N_G^-(v)$  the *outneighborhood* and *inneighborhood* of  $v$ , respectively, and write  $d_G^+(v) = |N_G^+(v)|$  and  $d_G^-(v) = |N_G^-(v)|$ . We will omit the subscript  $G$  whenever there is no risk of confusion. We will denote the minimum outdegree by  $\delta^+(G)$  and the minimum indegree by  $\delta^-(G)$ . We also set  $\delta^\pm(G) = \min\{\delta^+(G), \delta^-(G)\}$ . Finally we write  $a = (b \pm c)$  for  $a \in (b - c, b + c)$ .

For the sake of simplicity and clarity of presentation, and in order to shorten some of our proofs, no real effort has been made here to optimize the constants appearing in our results. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that the underlying parameter  $n$  is sufficiently large.

## 2 Tools

In this section we introduce the main tools to be used in the proofs of our results.

### 2.1 Probabilistic tools

We will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see [3], [15]).

**Lemma 2.1** *If  $X \sim \text{Bin}(n, p)$ , then*

- $\Pr(X < (1 - a)np) < e^{-a^2 np/2}$  for every  $a > 0$ ;
- $\Pr(X > (1 + a)np) < e^{-a^2 np/3}$  for every  $0 < a < 3/2$ .

### 2.2 $r$ -factors

One of the main ingredients in our results is the ability to find an  $r$ -factor in a graph with  $r$  as large as possible. The following theorem of Katerinis [16] shows that a dense graph contains a dense  $r$ -factor.

**Lemma 2.2** *Let  $r$  be a positive integer and let  $G$  be a graph such that:*

- (i)  $r|V(G)|$  is even, and
- (ii)  $\delta(G) \geq |V(G)|/2$ , and
- (iii)  $|V(G)| \geq 4r - 5$ .

*Then  $G$  contains an  $r$ -factor.*

When a given graph  $G$  is almost regular, it turns out that  $G$  contains  $r$ -factors with  $r$  much closer to  $\delta(G)$  than given by Lemma 2.2. The following lemma was proved by Christofides, Kühn and Osthus in [5].

**Lemma 2.3** *Let  $G$  be a graph on  $n$  vertices of minimum degree  $\delta = \delta(G) \geq n/2$ .*

- (i) *Let  $r$  be an even number such that  $r \leq \frac{\delta + \sqrt{n(2\delta - n)}}{2}$ . Then  $G$  contains an  $r$ -factor.*
- (ii) *Let  $0 < \xi < 1/9$  and suppose  $(1/2 + \xi)n \leq \Delta(G) \leq \delta + \xi^2 n$ . If  $r$  is an even number such that  $r \leq \delta - \xi n$  and  $n$  is sufficiently large, then  $G$  contains an  $r$ -factor.*

The result of Lemma 2.3 (ii) immediately implies the following useful corollary:

**Corollary 2.4** *Let  $1/2 < c \leq 1$  and let  $0 < \varepsilon < 1/9$  be such that  $c - \varepsilon - \sqrt{\varepsilon} \geq 1/2$ . Then for every sufficiently large integer  $n$  the following holds. Suppose that:*

- (i)  *$G$  is a graph with  $|V(G)| = n$ , and*
- (ii)  *$d(v) = (c \pm \varepsilon)n$  for every  $v \in V(G)$ .*

*Then  $G$  contains an  $r$ -factor for every even  $r \leq (c - \varepsilon')n$ , where  $\varepsilon' = \sqrt{\varepsilon} + \varepsilon$ .*

## 2.3 Permanent estimates

Let  $S_n$  be the set of all permutations of the set  $[n]$ . Given a permutation  $\sigma \in S_n$ , let  $A(\sigma)$  be an  $n \times n$  matrix which represents the permutation  $\sigma$ , that is, for every  $1 \leq i, j \leq n$ ,  $A(\sigma)_{ij} = 1$  if  $\sigma(i) = j$  and 0 otherwise. Notice that for every  $\sigma \in S_n$ , in each row and each column of  $A(\sigma)$  there is exactly one “1”. Every permutation  $\sigma \in S_n$  has a (unique up to the order of cycles) cyclic form. Given two  $n \times n$  matrices  $A$  and  $B$ , we write  $A \geq B$  in case that  $(A)_{ij} \geq (B)_{ij}$  for every  $1 \leq i, j \leq n$ . The *permanent* of an  $n \times n$  matrix  $A$  is defined as  $\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$ . Notice that in case  $A$  is a 0-1 matrix, every summand in the permanent is either 0 or 1, and the permanent of  $A$  counts the number of distinct permutations  $\sigma \in S_n$  which are *contained* in  $A$ , that is, the number of  $\sigma$ 's for which  $A \geq A(\sigma)$ . A 0-1 matrix  $A$  is called  *$r$ -regular* if it contains exactly  $r$  1's in every row and in every column.

Using the following two well known permanent estimates, in the next subsection we prove that if  $A$  is any 0-1  $\alpha n$ -regular matrix, then most of the permutations which are contained in it have relatively few cycles in their cyclic form.

We state first an upper bound for the permanent. This bound was conjectured by Minc and has been proven by Brégman [4].

**Theorem 2.5** *Let  $A$  be an  $n \times n$  matrix of 0-1 with  $t$  ones altogether. Then  $\text{per}(A) \leq \prod_{i=1}^n (r_i!)^{1/r_i}$ , where  $r_i$  are integers satisfying  $\sum_{i=1}^n r_i = t$  and are as equal as possible.*

A square matrix  $A$  of nonnegative real numbers is called doubly stochastic if each row and column of  $A$  sum to 1. The following lower bound is also known as the Van der Waerden conjecture and has been proven by Egorychev [9] and by Falikman [10]:

**Theorem 2.6** *Let  $A$  be an  $n \times n$  doubly stochastic matrix. Then  $\text{per}(A) \geq \frac{n!}{n^n}$ .*

## 2.4 2-factors with few cycles

Motivated by ideas from [2, 12, 20], in this subsection we prove that for every sufficiently large integer  $n$ , in every  $r$ -regular, 0-1,  $n \times n$  matrix  $A$ , most of the permutations contained in  $A$  have relatively few cycles in their cyclic form, provided that  $r$  is linear in  $n$ . We conclude that in every dense  $r$ -regular graph  $G$ , most of the 2-factors do not contain too many cycles (we consider a single edge as a cycle too). We also prove that in case  $r$  is even,  $G$  contains such a 2-factor with all cycles of length at least 3. These lemmas are crucial since one of the main ingredients of our proofs is the ability to find “enough” 2-factors with only few cycles and then to turn them into Hamilton cycles.

**Lemma 2.7** *For every  $\alpha > 0$  there exists an integer  $n_0 := n_0(\alpha)$  such that the following holds. Suppose that:*

- (i)  *$A$  is an  $n \times n$  matrix with  $n \geq n_0$ , and*
- (ii) *all entries of  $A$  are 0 or 1, and*
- (iii)  *$A$  is  $\alpha n$ -regular.*

*Then the number of permutations  $\sigma \in S_n$  for which  $A \geq A(\sigma)$  and such that there are at most  $s^* = \sqrt{n \ln n}$  cycles in their cyclic form, is  $(1 + o(1))^n \left(\frac{\alpha n}{e}\right)^n$ .*

Notice that in case  $A$  is the adjacency matrix of a graph  $G$ , every permutation  $\sigma \in S_n$  for which  $A \geq A(\sigma)$  corresponds to a 2-factor with exactly the same number of cycles as in the cyclic form of  $\sigma$  (we consider a single edge as a cycle too); and every 2-factor  $F$  of  $G$  corresponds to at most  $2^s$  permutations, where  $s$  is the number of cycles in  $F$  (each cycle can be oriented in at most two ways). Therefore, the following is an immediate corollary of Lemma 2.7:

**Corollary 2.8** *For every  $\alpha > 0$  there exists an integer  $n_0 := n_0(\alpha)$  such that the following holds. Suppose that:*

- (i)  *$G$  is a graph with  $|V(G)| = n \geq n_0$ , and*
- (ii)  *$G$  is  $\alpha n$ -regular.*

*Then the number of 2-factors of  $G$  with at most  $s^* = \sqrt{n \ln n}$  cycles is  $(1 + o(1))^n \left(\frac{\alpha n}{e}\right)^n$ .*



**Proof of Lemma 2.7.** Given a 0-1 matrix of order  $n \times n$ , let  $S(A) = \{\sigma \in S_n : A \geq A(\sigma)\}$  be the set of all permutations contained in  $A$ , and let  $f(A, k)$  be the number of permutations  $\sigma \in S(A)$  with exactly  $k$  cycles. Notice that  $f(A) := \sum_k f(A, k) = |S(A)|$ . Given an integer  $1 \leq t \leq n$  we also define

$$\phi(A, t) := \max\{f(A') : A' \text{ is a } t \times t \text{ submatrix of } A\}.$$

For the upper bound, using Theorem 2.5 and the fact that  $(k/e)^k \leq k! \leq k(k/e)^k$  we conclude that

$$\text{per}(A) \leq ((\alpha n)!)^{n/(\alpha n)} = (1 + o(1))^n \left(\frac{\alpha n}{e}\right)^n.$$

Now, notice that

$$\text{per}(A) = \sum_{s=1}^n f(A, s).$$

Applying Theorem 2.6 to the doubly stochastic matrix  $\frac{1}{\alpha n}A$ , we obtain:

$$\sum_{s=1}^n f(A, s) = \text{per}(A) \geq n! \alpha^n \geq \left(\frac{\alpha n}{e}\right)^n.$$

In order to complete the proof we need to show that  $\sum_{s > s^*} f(A, s) = o\left(\left(\frac{\alpha n}{e}\right)^n\right)$ . Let  $s > \sqrt{n \ln n}$ , we wish to estimate  $f(A, s)$  from above. Given a permutation  $\sigma \in S(A)$  with  $s$  cycles, there must be at least  $\frac{1}{2}\sqrt{n \ln n}$  cycles, each of which is of length at most  $2\sqrt{n/\ln n}$ . Therefore, by the pigeonhole principle we get that there must be a cycle length  $\ell := \ell(\sigma)$  which appears at least  $j = \frac{\ln n}{4}$  times in  $\sigma$ . The number of permutations in  $S(A)$  with  $s$  cycles which contain at least  $j$  cycles of fixed length  $\ell$  is at most:

$$\binom{n}{j} \prod_{i=1}^j (\alpha n)^{\ell-1} \cdot \phi(A, n - j\ell) \leq \left(\frac{en}{j}\right)^j (\alpha n)^{j\ell-j} \cdot \phi(A, n - j\ell). \quad (2.1)$$

Indeed, first we fix  $j$  cycles of length  $\ell$ . To do so we choose  $j$  elements,  $x_1, \dots, x_j$ , one for each such a cycle. This can be done in  $\binom{n}{j}$  ways. Since  $A$  is  $\alpha n$ -regular, for each  $1 \leq i \leq j$ , there are at most  $(\alpha n)^{\ell-1}$  options to close a cycle of length  $\ell$  which contains  $x_i$ . Given these  $j$  cycles of total length  $j\ell$ , there are at most  $\phi(A, n - j\ell)$  ways to extend it to a cyclic form of a permutation  $\sigma \in S(A)$ .

Next we estimate  $\phi(A, n - j\ell)$ . Let  $t = j\ell$  and let  $A_1$  be an arbitrary  $(n - t) \times (n - t)$  submatrix of  $A$ . By switching order of some rows and columns, we can assume that  $A = \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix}$ , where  $A_2$ ,  $B$  and  $C$  are  $t \times t$ ,  $(n - t) \times t$  and  $t \times (n - t)$  submatrices of  $A$ , respectively. Given a 0-1 matrix  $M$ , let  $g(M) = \mathbf{1}^T M \mathbf{1}$  be the number of 1's in  $M$  ( $\mathbf{1}$  is a vector with all entries equal 1). Since  $g(A_2) \leq t^2$  and since  $A$  is  $\alpha n$ -regular, it follows that  $g(B) \geq \alpha n t - t^2$ . Therefore, we conclude that  $g(A_1) = \alpha n(n - t) - g(B) \leq \alpha n(n - t) - (\alpha n t - t^2)$  and that the average number of 1's in a row or a column of  $A_1$  is

$$\frac{g(A_1)}{n - t} \leq \alpha n - \frac{t(\alpha n - t)}{n - t} =: d_1.$$



Note that  $\alpha(n-t) \leq d_1 \leq \alpha n$ . Now, by Brégman's Theorem 2.5 we get that

$$\begin{aligned} \text{per}(A_1) &\leq (d_1!)^{\frac{n-t}{d_1}} \leq \left( \left( \frac{d_1}{e} \right)^{d_1} d_1 \right)^{\frac{n-t}{d_1}} \leq \left( \frac{\alpha n - \frac{t(\alpha n - t)}{n-t}}{e} \right)^{n-t} (\alpha n)^{1/\alpha} \\ &\leq \left( \frac{\alpha n}{e} \right)^{n-t} \left( 1 - \frac{t(\alpha n - t)}{\alpha n(n-t)} \right)^{n-t} (\alpha n)^{1/\alpha} \leq \left( \frac{\alpha n}{e} \right)^{n-t} e^{-t+t^2/(\alpha n)} (\alpha n)^{1/\alpha}. \end{aligned}$$

Hence, we conclude that

$$\phi(A, n-t) \leq \left( \frac{\alpha n}{e} \right)^{n-t} e^{-t+t^2/(\alpha n)} (\alpha n)^{1/\alpha}.$$

Now, plugging it into the estimate (2.1) and recalling that  $\ell \leq 2\sqrt{n/\ln n}$ ,  $j = \frac{\ln n}{4}$  and  $t = j\ell \leq \frac{1}{2}\sqrt{n \ln n}$ , we have

$$\begin{aligned} f(A, s) &\leq \sum_{\ell \leq 2\sqrt{n/\ln n}} \left( \frac{en}{j} \right)^j (\alpha n)^{t-j} \phi(A, n-t) \\ &\leq \sum_{\ell \leq 2\sqrt{n/\ln n}} \left( \frac{en}{j} \right)^j (\alpha n)^{t-j} \left( \frac{\alpha n}{e} \right)^{n-t} e^{-t+t^2/(\alpha n)} (\alpha n)^{1/\alpha} \\ &\leq \left( \frac{\alpha n}{e} \right)^n \sum_{\ell \leq 2\sqrt{n/\ln n}} \left( \frac{en}{j} \right)^j (\alpha n)^{-j} e^{t^2/(\alpha n)} (\alpha n)^{1/\alpha} \\ &\leq \left( \frac{\alpha n}{e} \right)^n 2\sqrt{n/\ln n} \left( \frac{e/\alpha}{j} \right)^j e^{O(\ln n)} (\alpha n)^{1/\alpha} \\ &\leq \left( \frac{\alpha n}{e} \right)^n 2\sqrt{n/\ln n} n^{-\Omega(\ln \ln n)} n^{O(1)} (\alpha n)^{1/\alpha} \\ &= \left( \frac{\alpha n}{e} \right)^n \cdot o\left( \frac{1}{n} \right). \end{aligned}$$

This clearly implies that  $\sum_{s > s^*} f(A, s) = o\left( \left( \frac{\alpha n}{e} \right)^n \right)$  and completes the proof.  $\square$

In the following lemma we prove that, given a dense  $r$ -regular graph  $G$ , if  $r$  is even then  $G$  contains a 2-factor with not too many cycle, all of length at least 3.

**Lemma 2.9** *For every  $\alpha > 0$  there exists an integer  $n_0 := n_0(\alpha)$  such that the following holds. Suppose that:*

- (i)  $\alpha n$  is even, and
- (ii)  $G$  is a graph with  $|V(G)| = n$ , and

(iii)  $G$  is  $\alpha n$ -regular.

Then  $G$  contains a 2-factor with at most  $\sqrt{n \ln n}$  cycles, each of length at least 3.

**Proof.** Since  $\alpha n$  is even,  $G$  has an Eulerian orientation  $\vec{E}$  (if  $G$  is not connected, then find such an orientation for every connected component). Assume that  $V(G) = [n]$  and let  $A$  be an  $n \times n$  matrix of 0 and 1s such that  $a_{ij} = 1$  if and only if  $v_i v_j \in \vec{E}$ . Notice that  $A$  is an  $(\alpha n/2)$ -regular  $n \times n$  matrix, therefore by Lemma 2.7 we conclude that there exists a permutation  $\sigma \in S_n$  for which  $A \geq A(\sigma)$  with at most  $\sqrt{n \ln n}$  cycles in its cyclic form. Since every such permutation defines a 2-factor of  $G$ , and since each cycle is built by out-edges of the orientation  $\vec{E}$ , we conclude that the shortest possible cycle is of length at least 3.  $\square$

## 2.5 Rotations

The most useful tool in turning a path  $P$  into a Hamilton cycle is the Pósa rotation-extension technique (see [29]). Motivated by this technique, in this section we establish tools for turning a path into a Hamilton cycle under certain assumptions suitable for proving our main results.

First we need the following notation. Given a path  $P = v_0 v_1 \dots v_k$  in a graph  $G$  and a vertex  $v_i \in V(P)$ , define  $v_i^+ = v_{i+1}$  and  $v_i^- = v_{i-1}$  ( $v_0^- = v_k$  and  $v_k^+ = v_0$ ). For a subset  $I \subseteq V(P)$ , we define  $I^+ = \{v^+ : v \in I\}$  and  $I^- = \{v^- : v \in I\}$ .

Now, given a dense graph and a path in it, the following lemma enables us to obtain a longer path with only few rotations.

**Lemma 2.10** *Let  $G$  be a graph with  $|V(G)| = n$  and  $\delta(G) \geq n/2$ . Let  $P_0$  be a path in  $G$ . Then there exist two vertices  $a, b \in P_0$  and a path  $P^*$  in  $G$  connecting  $a$  to  $b$  so that:*

$$(i) \quad V(P^*) = V(P_0).$$

$$(ii) \quad |E(P_0) \Delta E(P^*)| \leq 4.$$

(iii)  $ab \in E(G)$  and the cycle obtained by adding this edge to  $P^*$  is a Hamilton cycle, or  $G$  contains an edge between  $\{a, b\}$  and  $V(G) \setminus V(P^*)$ .

**Proof.** Let  $P_0 = v_0 \dots v_k$  be a path in  $G$ . If there exists an edge  $v_0 v \in E(G)$  or  $v_k v \in E(G)$  for some  $v \notin V(P_0)$ , then by setting  $P_0 = P^*$ ,  $a = v_0$  and  $b = v_k$  we are done. Assume then that there is no such edge. In particular, it means that  $N(v_0) \cup N(v_k) \subseteq V(P_0)$ . First, we claim that there must be a vertex  $v \in N(v_0)^-$  such that  $vv_k \in E(G)$ . Otherwise, we have that  $N(v_k) \subseteq (V(P_0) \setminus \{v_k\}) \setminus N(v_0)^-$ . Since  $\delta(G) \geq n/2$  and since  $|V(P_0) \setminus \{v_k\}| \leq n - 1$  we conclude that  $|(V(P_0) \setminus \{v_k\}) \setminus N(v_0)^-| < n/2$  which is clearly a contradiction.

Let  $v_i \in N(v_0)^-$  be such vertex with  $v_i v_k \in E(G)$ . Notice that  $C = v_0 v_1 \dots v_i v_k v_{k-1} \dots v_{i+1} v_0$  is a cycle on the vertex set  $V(P_0)$ , obtained by deleting one edge from  $P$  and adding two new edges. If  $C$  is a Hamilton cycle then we are done. Otherwise, since  $G$  is a connected graph (this follows easily from  $\delta(G) \geq n/2$ ), there exist two vertices  $v \in V(C)$  and  $u \in V(G) \setminus V(C)$  such that  $vu \in E(G)$ . By deleting an edge  $vw$  from  $C$  and by denoting  $a = v$  and  $b = w$  we get the desired path.  $\square$

In the following lemma we prove that every dense graph  $G$  contains a subgraph  $H$  with some nice pseudorandom properties for which  $\text{reg}_{\text{even}}(G)$  and  $\text{reg}_{\text{even}}(G - H)$  are relatively close to each other. We will use this subgraph  $H$  to form edge disjoint Hamilton cycles from a set of edge disjoint 2-factors. It is crucial for the proof of Theorem 1.3.

**Lemma 2.11** *For every  $0 < \varepsilon < 1/4$  and  $0 < \alpha < \varepsilon^2$  there exists an integer  $n_0 := n_0(\varepsilon, \alpha)$  such that the following holds. Suppose that:*

- (i)  $G$  is a graph with  $|V(G)| = n \geq n_0$ , and
- (ii)  $\delta(G) \geq (1/2 + \varepsilon)n$ .

*Then  $G$  contains a subgraph  $H \subset G$  with the following properties:*

- (P1)  $G'$  is  $r$ -regular, where  $G' = G \setminus H$  and  $r$  is an even integer which satisfies  $r \geq (1 - \varepsilon/2)\text{reg}_{\text{even}}(G)$ ;
- (P2)  $\delta(H) \geq \varepsilon n/4$ ;
- (P3)  $|E_H(S, T)| \geq \beta n^2$  for every two disjoint subsets  $S, T \subseteq V(G)$  of size  $|S| = \alpha n$  and  $|T| \geq \frac{(1 - \varepsilon)n}{2}$ , where  $\beta > 0$  is some relatively small constant;
- (P4)  $|N_{H-E'}(S) \setminus S| \geq (1/2 + \varepsilon/4)n$  for every subset  $S \subset V(G)$ ,  $|S| = \alpha n$  and for every subset  $E' \subset E(H)$  of size  $|E'| = o(n^2)$ ;
- (P5)  $H - E'$  is a connected graph for every  $E' \subset E(H)$  such that  $\delta(H - E') \geq \alpha n$  and  $|E'| = o(n^2)$ .

**Proof of Lemma 2.11.** Let  $R$  be a  $\text{reg}_{\text{even}}(G)$ -factor of  $G$ . Observe that  $\text{reg}_{\text{even}}(G) \geq n/4$  by Lemma 2.2. Since  $\text{reg}_{\text{even}}(G)$  is even, we can find an Eulerian orientation  $\vec{E}$  and obtain an oriented graph  $\vec{R} = (V(G), \vec{E})$  (if  $G$  is not connected, then find such an orientation for every connected component). Now, choose a collection  $\mathcal{F}$  of  $\varepsilon n/8$  edge-disjoint random 1-factors from  $\vec{R}$  (the existence of such a collection follows immediately from the fact that  $\vec{R}$  is  $\text{reg}_{\text{even}}(G)/2$ -regular and from Theorem 2.6). Delete the orientation from edges in every  $F \in \mathcal{F}$  and let  $H$  be the graph spanned by the edges  $\cup_{F \in \mathcal{F}} E(F)$  and the edges of  $G - R$ . We prove that with high probability,  $H$  satisfies all the properties stated in the theorem.

Properties (P1) and (P2) follow immediately from the definition of  $H$  and from Theorem 2.2.

For proving (P3), let  $S, T \subseteq V(G)$  be two disjoint subsets for which  $|S| = \alpha n$  and  $|T| = \frac{(1 - \varepsilon)n}{2}$ . Since  $\delta(G) \geq (1/2 + \varepsilon)n$ , it follows that  $d(v, T) \geq \varepsilon n/2$  for every  $v \in S$ . Therefore,  $|E_G(S, T)| \geq |S|\varepsilon n/2 =$

$\frac{\varepsilon\alpha}{2}n^2$ . Now, if  $|E_{G-R}(S, T)| \geq \frac{\varepsilon\alpha}{4}n^2$ , then we are done. Otherwise,  $|E_R(S, T)| \geq \frac{\varepsilon\alpha}{4}n^2$ . We wish to bound from above the probability that for two such subsets, the 2-factors in  $H$  use at most  $\beta n^2$  edges from  $E_R(S, T)$ . For that, consider  $\vec{R}$  again and let  $A$  be an  $n \times n$ , 0/1 matrix for which  $(A)_{ij} = 1$  if and only if  $ij \in \vec{E}$ . Since  $A$  is  $\text{reg}_{\text{even}}(G)/2$ -regular, by Theorem 2.6 we conclude that

$$\text{per}(A) \geq \left( \frac{\text{reg}_{\text{even}}(G)}{2e} \right)^n.$$

Let  $A'$  be a matrix which is obtained from  $A$  by deleting  $cn^2/2$  many 1's (where  $c > 0$  is some positive constant). Then by Theorem 2.5 we get that

$$\text{per}(A') \leq (1 + o(1))^n \left( \frac{\text{reg}_{\text{even}}(G) - cn}{2e} \right)^n.$$

Now, picking a 1-factor  $F$  of  $\vec{R}$  at random, the probability that  $F \cup E_R(S, T)$  uses only edges from some fixed subset of edges between  $S$  and  $T$  of size at most  $\beta n^2$  is bounded from above by  $\frac{\text{per}(A')}{\text{per}(A)}$ , where  $c = 2\varepsilon\alpha/4 - 2\beta$  (recall that  $|E_R(S, T)| \geq \varepsilon\alpha n^2/4$ ). Note that when we remove a 1-factor from  $\vec{R}$ , the new graph remains regular (the degree decreases by exactly 1). Therefore, by the above arguments we conclude that the probability that there is a set of  $\beta n^2$  edges from  $E_R(S, T)$  such that all the 2-factors in  $\mathcal{F}$  use only edges from this set between  $S$  and  $T$  is at most

$$(1 + o(1))^{nt} \binom{n^2}{\beta n^2} \cdot \prod_{i=1}^t \left( \frac{\text{reg}_{\text{even}}(G) - cn - i}{\text{reg}_{\text{even}}(G) - i} \right)^n \leq (1 + o(1))^{nt} \left( \frac{e}{\beta} \right)^{\beta n^2} \delta^{nt} = \delta^{\Theta(n^2)}$$

where  $t = \varepsilon n/8$ ,  $\delta < 1$  and  $\beta$  is small enough. Applying the union bound we get that the probability for having two such sets is at most  $4^n \cdot \delta^{\Theta(n^2)} = o(1)$ .

Property (P4) follows immediately from (P3) and from the fact that  $|S| = \alpha n \leq \varepsilon^2 n \leq \varepsilon n/4$ . Indeed, given a subset  $S \subset V(G)$  for which  $|S| = \alpha n$ , by (P3) we get that the number of edges (in  $H$ ) between  $S$  to every subset of size  $(1/2 - \varepsilon/2)n$  is  $\Theta(n^2)$ . Therefore, by removing  $o(n^2)$  edges we cannot delete all the edges between two such sets. It follows that  $|N_{H-E'}(S) \setminus S| \geq (1/2 + \varepsilon/2 - \alpha)n \geq (1/2 + \varepsilon/4)n$ .

For (P5), notice that from the minimum degree condition we have that every component of  $H - E'$  has size at least  $\alpha n$ . Now, by (P4) we have that every connected component is in fact of size more than  $n/2$  even after deleting  $o(n^2)$  many edges. This completes the proof.  $\square$

In the next lemma, using some ideas from [31], we prove that in a graph with good expansion properties, every non-Hamilton path can be extended by changing only a few edges.

**Lemma 2.12** *For every  $0 < \varepsilon < 1/80$  there exists an integer  $n_0 := n_0(\varepsilon)$  such that the following holds. Suppose that:*

- (1)  $H$  is a graph with  $|V(H)| = n \geq n_0$ , and
- (2)  $\delta(H) \geq \varepsilon n/4$ , and
- (3)  $|N_H(S) \setminus S| > (1/2 + \varepsilon/4)n$  for every subset  $S \subset V(G)$  of size  $|S| = \varepsilon^3 n$ .

Then for every path  $P$  with  $V(P) \subseteq V(H)$  ( $P$  does not necessarily need to be a subgraph of  $H$ ), there exist a pair of vertices  $a, b$  and a path  $P^*$  in  $H \cup P$  connecting these vertices so that:

(i)  $V(P^*) = V(P)$ , and

(ii)  $|E(P) \Delta E(P^*)| \leq 8$ , and

(iii)  $ab \in E(H)$  and the cycle obtained by adding this edge is a Hamilton cycle, or  $H \cup P$  contains an edge between  $\{a, b\}$  and  $V(H) \setminus V(P^*)$ .

**Proof.** Let  $P = v_0 v_1 \dots v_k$  be a path. We distinguish between three cases:

**Case I:** There exists  $v \in V(H) \setminus V(P)$  for which  $v_0 v \in E(H)$  or  $v_k v \in E(H)$ . In this case, by denoting  $P^* = P$ ,  $a = v_0$  and  $b = v_k$ , we are done.

**Case II:**  $v_0 v_k \in E(H)$ . Let  $C$  be the cycle obtained by adding the edge  $v_0 v_k$  to  $P$ . If  $C$  is a Hamilton cycle then we are done. Otherwise, since  $H$  is connected (immediate from properties (2) and (3)), we can find  $v \in V(C)$  and  $u \in V(H) \setminus V(C)$  for which  $vu \in E(H)$ . Now, let  $P^*$  be the path obtained from  $C$  by deleting the edge  $vv^+$ ,  $a = v$ ,  $b = v^+$  and we are done.

**Case III:**  $N_H(v_0) \cup N_H(v_k) \subseteq V(P)$  and  $v_0 v_k \notin E(H)$ . Let  $t = 10/\varepsilon$  and let  $I_1, \dots, I_t$  be a partition of  $P$  into  $t$  intervals of length at most  $|P|/t \leq \varepsilon n/10$  each. By the minimum degree condition (Property (2)) and by the pigeonhole principle, we can find  $I_i \in \mathcal{F}$  for which  $|N_H(v_0) \cap I_i| \geq (\varepsilon n/4)/(10/\varepsilon) = \varepsilon^2 n/40$ . Similarly there exists an interval  $I_j$  which contains at least  $\varepsilon^2 n/40$  neighbors of  $v_k$ . If  $i \neq j$  then set  $I = I_i$  and  $J = I_j$ . Otherwise, divide  $I_i$  into two intervals such that each of them contains at least  $\varepsilon^2 n/80$  neighbors of  $v_0$ . Clearly one of them contains at least  $\varepsilon^2 n/80$  neighbors of  $v_k$ . Hence, we obtain two disjoint intervals  $I$  and  $J$  of  $P$  such that  $|I|, |J| \leq \varepsilon n/10$  and for which  $|N_H(v_0) \cap I|, |N_H(v_k) \cap J| \geq \varepsilon^2/80 \geq \varepsilon^3 n$ .

Now, assume that the interval  $I$  is to the left of the interval  $J$  according to the orientation of  $P$  (the case where  $I$  is to the right of  $J$  is similar). Let  $i_1 = \min\{i : v_i \in I\}$  and define  $L := \{v_0, \dots, v_{i_1-1}\}$  to be the set of all vertices of  $P$  which are to the left of  $I$ . For  $i_2 = \max\{i : v_i \in I\}$  and  $i_3 = \min\{i : v_i \in J\}$ , set  $M := \{v_{i_2+1}, \dots, v_{i_3-1}\}$  to be the set of all vertices between  $I$  and  $J$ . Similarly, set  $R := \{v_{i_4+1}, \dots, v_k\}$  to be the set of all vertices which are to the right of  $J$  in  $P$  (where  $i_4 = \max\{i : v_i \in J\}$ ). We prove that by a sequence of at most four additions and at most three deletions of edges we can turn  $P$  into a cycle  $C$  on  $V(P)$ , and then the result follows exactly as described in Case II (deleting at most one more edge). Let  $I_0 \subseteq N_H(v_0) \cap I$  and  $J_0 \subseteq N_H(v_k) \cap J$  be two subsets of size exactly  $\varepsilon^3 n$ . Let

$$N := (N_H(I_0^-)^+ \cap L) \cup (N_H(I_0^-)^- \cap M) \cup (N_H(I_0^-)^+ \cap R).$$

Then, by Property (3) we have  $|N| \geq (1/2 + \varepsilon/4)n - |I| - |J| > n/2$  and also  $|N_H(J_0^+)| > n/2$ . Therefore we conclude that  $N \cap N_H(J_0^+) \neq \emptyset$  and need to consider only the following three scenarios:

(a)  $(N_H(I_0^-)^+ \cap L) \cap N_H(J_0^+) \neq \emptyset$ . Let  $v^+ \in (N_H(I_0^-)^+ \cap L)$  and  $u^+ \in J_0^+$  be such that  $v^+ u^+ \in E(H)$ , and let  $w \in I_0$  be such that  $w^- v \in E(H)$ . Then we have the following cycle

$$C = v^+ \dots w^- v \dots v_0 w \dots u v_k \dots u^+ v^+.$$

(b)  $(N_H(I_0^-)^- \cap M) \cap N_H(J_0^+) \neq \emptyset$ . Let  $v^- \in (N_H(I_0^-))^- \cap M$  and  $u^+ \in J_0^+$  be such that  $v^- u^+ \in E(H)$ , and let  $w \in I_0$  be such that  $w^- v \in E(G)$ . In this case the cycle is

$$C = v \dots w v_k \dots u^+ v^- \dots w v_0 \dots w^- v.$$

(c)  $(N_H(I_0^-)^+ \cap R) \cap N_H(J_0^+) \neq \emptyset$ . Let  $v^+ \in (N_H(I_0^-)^+ \cap R)$  and  $u^+ \in J_0^+$  be such that  $v^+ u^+ \in E(H)$ , and let  $w \in I_0$  be such that  $w^- v \in E(H)$ . We obtain the following cycle

$$C = v_0 \dots w^- v \dots u^+ v^+ \dots v_k u \dots w v_0.$$

This completes the proof.  $\square$

## 2.6 Oriented graphs

In this subsection we establish tools needed in the proof of Theorem 1.7 which deals with counting the number of Hamilton cycles in oriented graphs. We start with the following notion of a *robust expander* due to Kühn, Osthus and Treglown [24]:

**Definition 2.13** *Let  $G$  be an oriented graph of order  $n$  and let  $S \subseteq V(G)$ . The  $\nu$ -robust outneighborhood  $RN_{\nu,G}^+(S)$  of  $S$  is the set of vertices with at least  $\nu n$  inneighbors in  $S$ . The graph  $G$  is called a robust  $(\nu, \tau)$ -outexpander if  $|RN_{\nu,G}^+(S)| \geq |S| + \nu n$  for every  $S \subseteq V(G)$  with  $\tau n \leq |S| \leq (1 - \tau)n$ .*

The following fact is an immediate consequence of the definition of a robust  $(\nu, \tau)$ -outexpander.

**Fact 2.14** *For every  $\nu > 0$  and for every  $\nu' < \nu$  there exists an integer  $n_0 := n_0(\nu, \nu')$  such that the following holds. Suppose that:*

- (i)  *$G$  is an oriented graph on  $n \geq n_0$  vertices.*
- (ii)  *$G$  is a robust  $(\nu, \tau)$ -outexpander.*

*Then every graph  $G'$  which is obtained from  $G$  by adding a new vertex (does not matter how) is a robust  $(\nu', \tau)$ -outexpander.*

The following theorem shows that given a robust outexpander  $G$  which is almost regular,  $G$  contains an  $r$ -factor with almost the same degree as the degrees of  $G$ .

**Theorem 2.15** *For positive constants  $\nu, \tau$  and  $\alpha$ , and a sufficiently large integer  $n$  such that  $0 < 1/n \ll \nu \leq \tau \ll \alpha < 1$ , and for every  $\eta > 0$  there exist  $\gamma = \gamma(\alpha, \nu, \tau, \eta) > 0$  such that the following holds. Suppose that:*

- (i)  *$G$  is an oriented graph with  $|V(G)| = n$ , and*

(ii)  $d^\pm(v) = (\alpha \pm \gamma)n$  for every  $v \in V(G)$ , and

(iii)  $G$  is a robust  $(\nu, \tau)$ -expander.

Then  $G$  contains an  $(\alpha - \eta)n$ -factor.

**Remark:** The proof of Theorem 2.15 follows immediately from Lemma 5.2 of [22] (which is proven using the Max-Flow-Min-Cut theorem) and the simple argument from the first paragraph of the proof of Corollary 1.2 in [28], so we omit the details.

The following technical lemma is one of the main ingredients in the proof of Theorem 1.7. We use it to turn a directed path of length  $n - o(n)$  into a directed Hamilton cycle:

**Lemma 2.16** *For every  $\alpha > 3/8$  there exists an integer  $n_0 := n_0(\alpha)$  such that the following holds. Suppose that:*

(i)  $G$  is an oriented graph with  $|V(G)| = n \geq n_0$ , and

(ii)  $\delta^\pm(G) \geq \alpha n$ .

Then for every two disjoint subsets  $A, B \subseteq V(G)$  with  $|A| = |B| = \alpha n/2$ ,  $G$  contains a Hamilton path which starts inside  $A$  and ends inside  $B$ .

Before proving Lemma 2.16 we need the following two results which are stated below. The first lemma, due to Kühn and Osthus [22], asserts that a dense oriented graph is also a robustly expanding graph.

**Lemma 2.17 (Lemma 12.1 [22])** *Let  $0 < 1/n \ll \nu \ll \tau \leq \varepsilon/2 \leq 1$  and suppose that  $G$  is an oriented graph on  $n$  vertices with  $\delta^+(G) + \delta^-(G) + \delta(G) \geq 3n/2 + \varepsilon n$ . Then  $G$  is a robust  $(\nu, \tau)$ -outexpander.*

The following theorem states that if a graph  $G$  is a robust outexpander with a linear minimum degree, then  $G$  contains a Hamilton cycle.

**Theorem 2.18 (Theorem 16 [24])** *Let  $n_0$  be a positive integer and  $\nu, \tau, \eta$  be positive constants such that  $1/n_0 \ll \nu \leq \tau \ll \eta < 1$ . Let  $G$  be a digraph on  $n \geq n_0$  vertices with  $\delta^\pm(G) \geq \eta n$  which is a robust  $(\nu, \tau)$ -outexpander. Then  $G$  contains a Hamilton cycle.*

Now we are ready to prove Lemma 2.16.

**Proof of Lemma 2.16.** Let  $\alpha > 3/8$  and let  $G$  be an oriented graph with  $|V(G)| = n \geq n_0$  and  $\delta^\pm(G) \geq \alpha n$ . Let  $A, B \subseteq V(G)$  be two disjoint subsets of size  $|A| = |B| = \alpha n/2$ . We wish to show that  $G$  contains a Hamilton path which starts inside  $A$  and ends inside  $B$ . First, notice that since



$\delta^-(G) + \delta^+(G) + \delta(G) \geq 3n/2 + \varepsilon n$  (for some small positive constant  $\varepsilon$ ), by Lemma 2.17 we get that for every choice of constants  $0 < \nu < \tau \leq \varepsilon/2$ ,  $G$  is a robust  $(\nu, \tau)$ -outexpander. Second, by adding a new vertex  $x$  to  $V(G)$  in such a way that  $N^+(x) = A$  and  $N^-(x) = B$ , by Fact 2.14 we obtain a new graph  $G'$  which is a robust  $(\nu/2, \tau)$ -outexpander. Third, by applying Theorem 2.18 to  $G'$  (applied with  $\eta = \alpha/2$ ), we conclude that  $G'$  is Hamiltonian. Last, let  $C$  be a Hamilton cycle in  $G'$ , by deleting  $x$  we obtain the desired Hamilton path in  $G$ .  $\square$

The following lemma enables us to pick a subgraph of an oriented graph which inherits some properties of the base graph.

**Lemma 2.19** *For every  $c > 0$ , for every  $0 < \varepsilon < c/2$ , and for every sufficiently large integer  $n$  the following holds. Suppose that:*

- (i)  $G$  is an oriented graph with  $|V(G)| = n$ , and
- (ii)  $d^\pm(v) = (c \pm \varepsilon)n$  for every  $v \in V(G)$ .

*Then there exists a subset  $V_0 \subseteq V(G)$  for which the following two properties hold:*

(P1)  $|V_0| = \Theta(n^{2/3})$ .

(P2)  $|N_G^+(v) \cap V_0| \in (c \pm 2\varepsilon)|V_0|$  and  $|N_G^-(v) \cap V_0| \in (c \pm 2\varepsilon)|V_0|$  for every  $v \in V(G)$ .

**Proof.** Let  $V_0$  be a subset of  $V(G)$  obtained by adding each  $v \in V(G)$  to  $V_0$  with probability  $p = n^{-1/3}$ , independently at random. We prove that  $V_0$  typically satisfies properties (P1) and (P2).

For (P1), notice that since  $|V_0| \sim \text{Bin}(n, p)$ , it follows by Lemma 2.1 that  $\Pr(|V_0| > 2np) \leq e^{-np/3} = o(1)$  and that  $\Pr(|V_0| < np/2) \leq e^{-np/8} = o(1)$ .

In order to prove (P2), let  $v \in V(G)$  be an arbitrary vertex. Since  $|N_G^+(v) \cap V_0| \sim \text{Bin}(d^+(v), p)$  and since  $d^+(v) \in (c \pm \varepsilon)n$ , by Lemma 2.1 (Chernoff) we have that  $\Pr(|N_G^+(v) \cap V_0| \geq (c + 2\varepsilon)n) \leq e^{-anp}$ , for some positive constant  $a = a(\varepsilon)$ . Applying the union bound we get that

$$\Pr\left(\exists v \in V(G) \text{ such that } |N_G^+(v) \cap V_0| \geq (c + 2\varepsilon)n\right) \leq ne^{-anp} = ne^{-an^{2/3}} = o(1).$$

In a similar way we prove it for  $|N_G^-(v) \cap V_0|$ . This completes the proof.  $\square$

Last, we need the following simple fact:

**Fact 2.20** *Let  $G$  be an oriented graph with  $|V(G)| = n$  and  $\delta^\pm(G) \geq 3n/8$ . Then, the directed diameter of  $G$  is at most 4.*

**Proof.** Let  $x, y \in V(G)$ . We wish to prove that there exists a path  $P$  of length at most 4 which is oriented from  $x$  to  $y$ . Let  $A \subseteq N_G^+(x)$  and  $B \subseteq N_G^-(y)$  two subsets of size  $|A| = |B| = 3n/8$ . If

$A \cap B \neq \emptyset$  then we are done. Otherwise, let  $a \in A$  be a vertex for which  $d^+(a, A) \leq |A|/2$  (there must be such a vertex since  $\sum_{z \in A} d^+(z, A) \leq \binom{|A|}{2}$ ), and let  $b \in B$  be a vertex for which  $d^-(b, B) \leq |B|/2$ . The result will follow by proving that  $N_G^+(a) \cap B \neq \emptyset$ ,  $N_G^+(a) \cap N_G^-(b) \neq \emptyset$  or  $N_G^-(b) \cap A \neq \emptyset$ . Indeed, otherwise we get that  $|V(G)| = n \geq 2 + |A| + |B| + |A|/2 + |B|/2 \geq 3n/8 + 3n/8 + 3n/16 + 3n/16 > n$ , which is a contradiction.  $\square$

### 3 Counting Hamilton cycles in undirected graphs

In this section we prove Theorem 1.4 and Corollary 1.5.

**Proof of Theorem 1.4.** Let  $H \subseteq G$  be a  $\text{reg}(G)$ -factor of  $G$ . By Lemma 2.2 we have that  $\text{reg}(G) = \Theta(n)$ . Therefore, we can apply Lemma 2.8 and conclude that  $\sum_{s \leq s^*} f(H, s) \geq \left(\frac{\text{reg}(G)}{e}\right)^n (1 - o(1))^n$  (where  $s^* = \sqrt{n \ln n}$  and  $f(H, s)$  counts then number of 2-factors of  $H$  with exactly  $s$  cycles).

Now, working in  $G$ , given a 2-factor  $F$  with  $s \leq s^*$  cycles, by applying Lemma 2.10 and using the connectivity of  $G$  we can turn  $F$  into a Hamilton cycle of  $G$  by adding and removing at most  $O(s)$  edges in the following way: let  $C$  be a non-Hamilton cycle in  $F$ . Since  $G$  is connected we can find a vertex  $v \in V(C)$  and a vertex  $u \in V(G) \setminus V(C)$  for which  $vu \in E(G)$ . Now, by deleting the edge  $vv^+$  from  $C$  we get a path  $P$  which can be extended by the edge  $vu$ . Connecting it to a cycle  $C'$  which contains  $u$  ( $C'$  can be just an edge) we obtain a longer path  $P'$ . Repeat this argument as long as we can. If there are no edges between the endpoints of the current path  $P'$  and other cycle from  $F$ , then we can use Lemma 2.10 in order to close  $P'$  to a cycle within 4 edge replacements. Then, using the connectivity of  $G$  again, we can extend it. In any case, we invest  $\Theta(1)$  edge replacements in order to decrease the number of cycles by 1, and unless the current cycle is a Hamilton cycle, we can always merge two cycles. Therefore, after  $O(s^*)$  edge-replacements we get a Hamilton cycle.

In order to complete the proof, notice that given a Hamilton cycle  $C$  in  $G$ , by replacing at most  $k$  edges we can get at most  $\binom{n}{k} (2k)^{2k}$  2-factors in  $H$  (choose  $k$  edges of  $C$  to delete, obtain at most  $k$  paths which need to be turned into a 2-factor by connecting endpoints of paths; for each endpoint we have at most  $2k$  choices of other endpoints to connect it to). Therefore, for some positive constant  $D$  we have that  $\sum_{s \leq s^*} f(G, s) \leq h(G) \cdot n \binom{n}{Ds^*} (2Ds^*)^{2Ds^*} \leq h(G) n^{O(s^*)}$ . This implies that

$$\begin{aligned} h(G) &\geq (1 - o(1))^n \left(\frac{\text{reg}(G)}{e}\right)^n n^{-O(s^*)} = (1 - o(1))^n \left(\frac{\text{reg}(G)}{e}\right)^n e^{-O(\sqrt{n} \ln^{3/2} n)} \\ &= (1 - o(1))^n \left(\frac{\text{reg}(G)}{e}\right)^n, \end{aligned}$$

and completes the proof of Theorem 1.4.  $\square$

Corollary 1.5 follows easily from Theorem 1.4.

**Proof of Corollary 1.5.** Let  $A$  be the adjacency matrix of  $G$ . Then  $A$  is an  $n \times n$  matrix with all entries 0's and 1's which is  $cn$ -regular (the number of 1's in each row/column is exactly  $cn$ ). Since

$G$  is  $cn$ -regular, the lower bound on  $h(G)$  follows from Theorem 1.4. For the upper bound, notice that since the number of Hamilton cycles in  $G$ ,  $h(G)$ , is at most the number of 2-factors in  $G$ , using Theorem 2.5 we get that

$$h(G) \leq \sum_{F\text{-2-factor in } G} 2^{c(F)} = \text{per}(A) \leq ((cn)!)^{1/c} = (1 + o(1))^n \left(\frac{cn}{e}\right)^n.$$

This completes the proof.  $\square$

## 4 Counting Hamilton cycles in oriented graphs

In this section we prove Theorem 1.7.

**Proof of Theorem 1.7.** Let  $c > 3/8$  and let  $\eta > 0$ . Let  $\varepsilon > 0$  be a positive constant for which  $3c > 3n/2 + \varepsilon n$ . Let  $\nu$  and  $\tau$  be positive constants such that  $0 < 1/n \ll \nu \ll \tau \leq \varepsilon/2 \leq 1$ . Applying Theorem 2.15 with respect to  $\alpha = c$  (using Lemma 2.17) we get a positive constant  $\gamma$  such that the following holds: every oriented graph  $G$  with  $|V(G)| = n$  which satisfies  $d^\pm(v) \in (c \pm \gamma)n$  for every  $v \in V(G)$ , contains a  $(c - \eta)n$ -factor.

Now, let  $G$  be an oriented graph which satisfies properties (i) and (ii) of the Theorem 1.7 with respect to  $\varepsilon' = \gamma/3$ . Applying Lemma 2.19 we get a subset  $V_0 \subset V(G)$  of size  $|V_0| = \Theta(n^{2/3})$  for which  $|N_G^+(v) \cap V_0| \in (c \pm 2\gamma/3)|V_0|$  and  $|N_G^-(v) \cap V_0| \in (c \pm 2\gamma/3)|V_0|$  for every vertex  $v \in V(G)$ . Let  $G_1 = G[V_0]$  and  $G_2 = G[V(G) \setminus V_0]$  denote the two subgraphs induced by  $V_0$  and  $V(G) \setminus V_0$ , respectively. Since we deleted at most  $o(n)$  vertices of  $G$  to obtain  $G_2$  it still satisfies properties (i) and (ii) of the theorem. Using the above mentioned arguments,  $G_2$  contains a  $(c - \eta)n'$ -factor  $H$ , where  $n' = |V(G_2)| = n - o(n)$ . Assume that  $V(G_2) = [n']$  and let  $A$  be an  $n' \times n'$  matrix with all entries 0's and 1's for which  $(A)_{ij} = 1$  if and only if  $ij \in E(H)$ .  $A$  is clearly  $(c - \eta)n'$ -regular, and therefore by Lemma 2.7 we get that there are at least  $\left(\frac{(c-\eta)n}{e}\right)^n (1 - o(1))^n$  permutations  $\sigma \in S_{n'}$  such that  $A \geq A(\sigma)$  and such that  $\sigma$  contains at most  $\sqrt{n \ln n}$  cycles in its cyclic form. Notice that every such permutation corresponds to a 1-factor of  $G_2$  with at most  $\sqrt{n \ln n}$  many cycles. Now, given a 1-factor  $F$  of  $G_2$ , we wish to turn it into a Hamilton cycle of  $G$  by changing at most  $O(n^{2/3})$  edges. This can be done as follows. Let  $C$  be a cycle in  $F$ . Since  $G_2$  is strongly connected (follows for example from Fact 2.20) we can find a vertex  $v \in V(C)$  and a vertex  $u \in V(G_2) \setminus V(C)$  for which  $vu \in E(G)$ . Deleting the edge  $vv^+$  from  $C$  we get a path  $Q$  which can be extended to a longer path  $Q'$  by adding the edge  $vu$  and all edges of the cycle  $C'$  in  $F$  including  $u$  apart from  $u^-u$ . Let  $x$  and  $y$  be the endpoints of the current path  $Q'$  (from  $x$  to  $y$ ). Using the subgraph  $G_1$ , we can close  $Q'$  to a cycle, using at most 6 additional edges. Indeed, by property Property (P2) of Lemma 2.19  $x$  has an inneighbor and  $y$  has an outneighbor in  $V_0$  and by Fact 2.20)  $y$  can be connected to  $x$  by a directed path of length at most 4. Delete from  $G_1$  the edges and vertices we used to close  $Q'$ . Update  $F$  by replacing  $C$  and  $C'$  by the newly created cycle. Repeat this argument until we have a cycle  $C$  with  $V(G_2) \subseteq V(C)$ . So far, we have invested  $O(\sqrt{n \ln n})$  edge replacements and have deleted at most  $O(\sqrt{n \ln n}) = o(|V_0|)$  vertices from  $G_1$ . Hence,  $G_1$  (minus all the edges/vertices deleted so far)

still satisfies (i) and (ii) of Lemma 2.16. Deleting an arbitrary edge  $vu$  from  $C$ , we obtain a path  $P$  with  $v, u$  as its endpoints. Next, choose disjoint sets  $A \subset N_G^+(v) \cap V_0$  and  $B \subset N_G^-(u) \cap V_0$ , each of size at least  $(c - \eta)|V_0|/2$ . Using (P2) of Lemma 2.19, and applying Lemma 2.16 with respect to  $A = N_G^+(v) \cap V_0$  and  $B = N_G^-(u) \cap V_0$  we obtain a Hamilton path  $P'$  of  $G_1$  which starts inside  $A$  and ends inside  $B$ . This path, together with  $P$  forms a Hamilton cycle of  $G$ . Note that this cycle was obtained from  $F$  by changing  $O(n^{2/3})$  edges.

In order to complete the proof, we need to show that for most of the 1-factors we get distinct Hamilton cycles. This can easily be obtained using similar calculations as in Theorem 1.4 which we omit.

For the upper bound on  $h(G)$ , observe that the number of Hamilton cycles in  $G$  is at most the number of 1-factors in  $G$ , and the latter can be estimated using Brégman's bound Theorem 2.5 as before.  $\square$

## 5 Packing Hamilton cycles in undirected graphs

In this section we prove Theorem 1.3.

**Proof of Theorem 1.3.** Let  $\varepsilon > 0$  and let  $G$  be a graph with minimum degree  $\delta(G) \geq (1/2 + \varepsilon)n$ . Let  $\varepsilon' < \min\{\varepsilon, 1/4\}$  be a positive constant, let  $H \subset G$  be an auxiliary subgraph of  $G$  obtained by applying Lemma 2.11 to  $G$  with  $\varepsilon'$ , and let  $G' = G - H$ . Recall that by property (P1) of Lemma 2.11  $G'$  is  $r$ -regular for some even integer  $r$  which satisfies  $r \geq (1 - \varepsilon/2)\text{reg}_{\text{even}}(G)$ . The result will then follow by proving that  $G$  contains at least  $(1 - \varepsilon/2)r/2$  edge disjoint Hamilton cycles.

First, notice that since  $\delta(G) > n/2$ , by Lemma 2.2 it follows that  $r = \Theta(n)$ . Therefore, we can use Lemma 2.9 repeatedly in order to find  $m = (1 - \varepsilon)r/2$  edge-disjoint 2-factors of  $G'$ ,  $\{F_1, \dots, F_m\}$ , each of them containing at most  $s^* = \sqrt{n \ln n}$  cycles. Now, we wish to turn each of the  $F_i$ s into a Hamilton cycle  $H_i$ , using the edges of  $G \setminus (H_1 \cup \dots \cup H_{i-1} \cup F_{i+1} \cup \dots \cup F_m)$ . For this goal, we make an extensive use of Lemma 2.12 and the properties of the auxiliary graph  $H$ .

Assume inductively that we have built edge-disjoint Hamilton cycles  $H_1, \dots, H_{i-1}$ , which are edge disjoint from  $F_i, \dots, F_m$ . Moreover, assume that each of the  $H_j$ 's has been created from  $F_j$  by replacing  $O(s^*)$  edges and that for  $G_i = G \setminus (H_1 \cup \dots \cup H_{i-1} \cup F_i \cup \dots \cup F_m)$  we have  $\delta(G_i) \geq \varepsilon n/4$ . Indeed, notice that  $G_0 = G \setminus (F_1 \cup \dots \cup F_m)$  contains  $H$  as a subgraph and  $G_i$  has the same minimum degree as  $G_0$ . Starting with  $F_i$ , using the connectivity of  $H$  and Lemma 2.12, we can turn  $F_i$  into a Hamilton cycle by using  $O(s^*)$  edge replacements (coming from  $H$ ), in a similar way as we did in the proof of Theorem 1.4. During the procedure, every edge that we delete from  $F_i$  is added back to  $H$  and therefore the minimum degree of  $H$  remains the same. Since this procedure takes  $O(s^*)$  edge replacements each time and since there are  $\Theta(n)$  factors to work on, the total number of edges deleted from the original  $H$  is at most  $O(ns^*) = o(n^2)$ . Thus by Properties (P4) and (P5) of Lemma 2.11, and since  $n$  is sufficiently large, the current version of  $H$ , obtained after adding and removing edges to the original  $H$  during this procedure, still satisfies the properties needed in order to apply Lemma 2.12 again. This completes the proof.  $\square$

## 6 Concluding remarks

We presented a general approach, based on permanent estimates, for counting and packing Hamilton cycles in dense graphs and oriented graphs. Using this method we derived some known results in a simpler way and proved some new results as well. In particular, we showed how to apply our technique to find many edge-disjoint Hamilton cycles in dense graphs.

It would be interesting to decide whether our approach can be also used to find many edge-disjoint Hamilton cycles in dense oriented graphs. The main obstacle here is that apparently there is no good analog of Pósa's rotation extension technique for digraphs.

In Theorem 1.4 we obtained a lower bound on  $h(G)$  in terms of  $\text{reg}_{\text{even}}(G)$ , for a Dirac graph  $G$ . For graphs which are not close to being regular our result is worse than the result of Cuckler and Kahn in [7]. It would be very interesting to try and approach their result using our method.

Another natural question is to obtain a variant of Theorem 1.7 for non-regular oriented graphs similar to the result of Cuckler and Kahn for the non-oriented case. The goal here is to estimate the minimum number of Hamilton cycles in an oriented graph on  $n$  vertices with semi-degree  $\delta^\pm(G) \geq (3/8 + o(1))n$ . Observe that our technique allows to prove easily that an oriented graph  $G$  on  $n$  vertices with  $\delta^\pm(G) \geq (3/8 + \varepsilon)n$  contains at least  $(\frac{\varepsilon n}{3e})^n$  Hamilton cycles. Indeed, applying repeatedly the result of Keevash, Kühn and Osthus [17] we can extract  $\frac{\varepsilon n}{2}$  edge-disjoint Hamilton cycles in such graph, whose union is an  $\frac{\varepsilon n}{2}$ -factor  $F$  in  $G$ . The rest of the proof is quite similar to our argument in Theorem 1.7. This establishes a weak(er) version of the result of Sárközy, Selkow and Szemerédi [30] for the oriented case.

Finally it would be also nice to extend the result of Keevash, Kühn and Osthus [17] and determine the number of edge disjoint Hamilton cycles that oriented graphs with  $\delta^\pm(G) \geq 3n/8$  must contain as a function of  $\delta^\pm(G)$ .

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